

## DSP: Lec 4

### Quiz

1] Prove that

$$Z[a^n x(n)] = X\left(\frac{z}{a}\right)$$

2] Deduce the z-transform for

a)  $x(n-2)$

b)  $x(n+1)$

1]

$$x(n) \longrightarrow X(z) = Z[x(n)]$$

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z[a^n x(n)] = \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= x(0) + \underbrace{a x(1)}_{\text{circled}} z^{-1} + a^2 x(2) z^{-2} + \dots$$

$$= x(0) + x(1) \left(\frac{z}{a}\right)^{-1} + x(2) \left(\frac{z}{a}\right)^{-2} + \dots$$

$$x(n) \longrightarrow X(z) = x(0) + x(1) z^{-1} + x(2) \left(\frac{z}{a}\right)^{-2} + \dots$$

$$\therefore X\left(\frac{z}{a}\right) = x(0) + x(1) \left(\frac{z}{a}\right)^{-1} + \dots$$

$$= Z[a^n x(n)]$$

2

$$a) \mathcal{Z}[x(n-2)] = \sum_{n=-\infty}^{\infty} x(n-2) z^{-n}$$

$$= x(-2) + x(-1) z^{-1} + x(0) z^{-2} + \dots$$

$$= x(-2) + x(-1) z^{-1} + z^{-2} [x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots]$$

$\underbrace{\hspace{10em}}_{\mathcal{Z}[x(z)]}$

$$= x(-2) + x(-1) z^{-1} + z^{-2} x(z)$$

$$b) \mathcal{Z}[x(n+1)] = \sum_{n=-\infty}^{\infty} x(n+1) z^{-n}$$

$$= x(1) + x(2) z^{-1} + x(3) z^{-2} + x(4) z^{-3} + \dots$$

$$= z \cdot z^{-1} [x(1) + x(2) z^{-1} + \dots]$$

$$= z [x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots]$$

$$= z \left[ \underbrace{x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots}_{\mathcal{Z}[x(z)]} - x(0) \right]$$

$$= z [x(z) - x(0)] = z x(z) - z x(0)$$

□ p. 4

$$c) \quad \mathcal{Z}[x(n+2)]$$

$$= \sum_{n=0}^{\infty} x(n+2) z^{-n}$$

$$= x(2) + x(3) z^{-1} + x(4) z^{-2} + \dots$$

$$= z^2 \cdot z^{-2} [x(2) + x(3) z^{-1} + x(4) z^{-2} + \dots]$$

$$= z^2 [x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + \dots]$$

$$= z^2 [x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots]$$

$$- \dots - x(0) - x(1) z^{-1}]$$

$$= z^2 [x(z) - x(0) - x(1) z^{-1}]$$

$$\boxed{\mathcal{Z}[x(n+2)] = z^2 x(z) - x(0) z^2 - x(1) z}$$

## \* Shifting Property:-

$$\begin{array}{l} \rightarrow \text{Delay} \quad x(n-a) \\ \rightarrow \text{Advance} \quad x(n+a) \end{array}$$

Find Z.T for  $x(n-1)$  Delay

$$x(n) \xrightarrow{\text{Z.T}} Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

$$Z[x(n-1)] = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n}$$

$$= x(-1) + x(0) z^{-1} + x(1) z^{-2} + \dots$$

$$= x(-1) + z^{-1} [x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots]$$

$$= x(-1) + z^{-1} X(z)$$

$$Z[x(n-2)] = \sum_{n=-\infty}^{\infty} x(n-2) z^{-n}$$

$$= x(-2) + x(-1) z^{-1} + x(0) z^{-2} + \dots$$

$$= x(-2) + x(-1) z^{-1} + z^{-2} [x(0) + x(1) z^{-1} + \dots]$$

$$Z[x(n-2)] = x(-2) + x(-1)z^{-1} + z^{-2}x(0)$$

$$x(n) \xrightarrow{Z.T.} x(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

assume  $x(0)=0, n < 0$

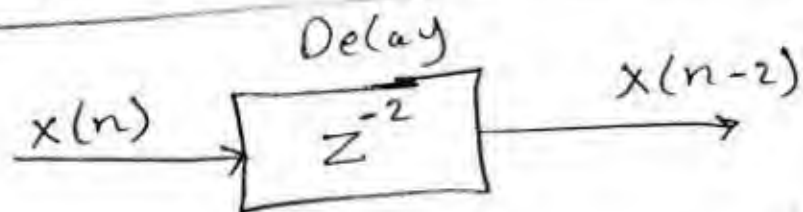
$$x(-1), x(-2), x(-3) \dots = 0$$

$$\therefore Z[x(n-2)] = z^{-2}x(z)$$

$$Z[x(n-1)] = z^{-1}x(z)$$

$$\therefore Z[x(n-m)] = z^{-m}x(z)$$

$$; x(z) = Z[x(n)]$$



# Advance

Find Z.T for  $x(n+1)$ :

$$Z[x(n+1)] = \sum_{n=0}^{\infty} x(n+1) z^{-n}$$

$$= x(1) + x(2) z^{-1} + x(3) z^{-2} + \dots$$

$$= z \cdot z^{-1} [x(1) + x(2) z^{-1} + \dots]$$

$$= z [x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots]$$

$$= z [x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots - x(0)]$$

$$= z [x(z) - x(0)] = z x(z) - z x(0)$$

$$* Z[x(n+1)] = z x(z) - z x(0)$$

$$Z[x(n+2)] = \sum_{n=0}^{\infty} x(n+2) z^{-n}$$

$$= x(2) + x(3) z^{-1} + x(4) z^{-2} + \dots$$

$$= z^2 \cdot z^{-2} [ \quad ]$$

$$= Z^2 [x(2) Z^{-2} + x(3) Z^{-3} + \dots - \dots]$$

$$= Z^2 [x(0) + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + \dots - x(0) - x(1) Z^{-1}]$$

$$= Z^2 [x(2) - x(0) - x(1) Z^{-1}]$$

$$\boxed{Z[x(n+2)] = Z^2 x(z) - x(0) Z^2 - x(1) Z}$$



← الحالة الـ هيجيبه الدكتور في الامتحان الـ (Delay) Difference

$$Z[x(n-m)] = Z^{-m} \cdot x(z)$$

Difference eqn. ← الحالة الـ

\* initial value

$$x(0) = x(n) \Big|_{n=0} = \lim_{z \rightarrow \infty} x(z)$$

\* Final value

$$x(\infty) = x(n) \Big|_{n=\infty} = \lim_{z \rightarrow 1} (z-1)x(z)$$

$$= \lim_{z \rightarrow 1} (1-z^{-1})x(z)$$

$(1-z^{-1})$  هي نفسها  $\frac{z-1}{z}$  في الحالة الأولى مكتوبه

$(z-1)$  بس لأنه المقام عبارة عن  $z$  و  $(z-1)$  فلا تؤثر.

\* Inverse Z-T  $[z^{-1}T]$

$$x(n) \xrightarrow{z \cdot T} x(z)$$

discrete time domain  $\xleftarrow[\text{Inverse}]{z^{-1} \cdot T}$  z. domain



$$\underline{[z^{-1} T] \text{ ال طريقة}}$$

① using [P.F] (closed form solution)

② using long division  
(open form solution)

sequence  $x(0), x(1), x(2), \dots$  ← open form

$x(n)$  ← closed

Ex

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Find  $x(n)$

Solu. using P.F

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

$$z^2 - 1.5z + 0.5 = (z - 1)(z - 0.5)$$

$$X(z) = Z \left[ \frac{z}{(z-1)(z-0.5)} \right]$$

$$= Z \left( \frac{A_1}{z-1} + \frac{A_2}{z-0.5} \right)$$

$$A_1 = \frac{1}{1-0.5} = 2 \quad A_2 = \frac{0.5}{-0.5} = -1$$

$$X(z) = Z \left[ \frac{2}{z-1} - \frac{1}{z-0.5} \right]$$

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

~~$$X(z) = Z^{-1} T \Rightarrow$$~~ 
$$x(n) = 2u(n) - (0.5)^n$$

$$x(n) = 2u(n) - (0.5)^n$$

← لو لم نكتب  $u(n)$  صح ولو كتبها صح

دورها انها تعرفني فيه (shift) ولا رأ.

**EX**

$$X(z) = \frac{1}{(z-1)(z-0.3)}$$

using Partial Fraction

$$X(z) = \frac{A_1}{z-1} + \frac{A_2}{z-0.3}$$

$$A_1 = \frac{1}{0.7} = 1.4 \quad A_2 = \frac{-1}{0.7} = -1.4$$

$$X(z) = \frac{1.4 \times z \times z^{-1}}{z-1} - \frac{1.4 \times z \times z^{-1}}{z-0.3}$$

$z^{-1}$   
(shift) بعل

$$z^{-1} T \Rightarrow X(n) = 1.4 u(n-1) - 1.4 (0.3)^{n-1} u(n-1)$$

در ها ال (shift)

و تقو ك ال الاله (0.3)<sup>n</sup> ليس لها وجود الا بعد القيه n=1

$$n=0 \Rightarrow X(0) = 0$$

$$X(0) = X(n) \Big|_{n=0} = \lim_{z \rightarrow \infty} X(z)$$

$$= \lim_{z \rightarrow \infty} \frac{1}{(z-1)(z-0.3)} = \frac{1}{\infty} = 0$$

ممكن تبينه باي قانون  
X(0)

**II** Lec 4

Ex

Midterm Previous  
Year

$$X(z) = \frac{z^3}{(z-1)(z^2-z+1)}$$

① Find  $x(0)$ ,  $x(1)$ ,  $x(2)$ , using long division method.

② Find  $x(n)$  using P.F.

→ or Find  $x(0)$ ,  $x(1)$ ,  $x(2)$  using P.F.   
  $x(2)$ ,  $x(1)$ ,  $x(0)$  ←  $x(n)$  هتجيب وبتجيبها قيم

$$(z-1)(z^2-z+1) \stackrel{\text{Sol}}{=} z^3 - z^2 + z - z^2 + z - 1$$

$$X(z) = \frac{z^3}{z^3 - 2z^2 + 2z - 1}$$

← القسمة الملهمة

[12] Lec 4

طلب منا  $X(2), X(1), X(0)$   
 يبقى لتجيب الثلاثة دول  
 وتقف

$$1 + 2z^{-1} + 2z^{-2}$$

$$z^3 : \text{ } \cancel{z^3}$$

$$z^3 - 2z^2 + 2z - 1$$

$$\frac{-z^3 + 2z^2 + 2z - 1}{z^3 - 2z^2 + 2z - 1}$$

$$2z^2 - 2z + 1$$

$$\frac{-2z^2 + 4z + 4 - 2z^{-1}}{2z^2 - 2z + 1}$$

$$2z - 3 + 2z^{-1}$$

$$X(z) = 1 + 2z^{-1} + 2z^{-2} + \dots$$

$$= X(0) + X(1)z^{-1} + X(2)z^{-2} + \dots$$

$$X(0) = 1, \quad X(1) = 2, \quad X(2) = 2$$

[2] using p.f

$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 1)}$$

$$= z \left[ \frac{z^2}{(z-1)(z^2 - z + 1)} \right]$$

$$X(z) = Z \left[ \frac{A}{z-1} + \frac{Bz+C}{z^2-z+1} \right]$$

$$A = \frac{1}{1-1+1} = 1$$

الحد الثاني هو مجموع المقامات ، نقارن المقامات .

$$B=0 \quad \text{و} \quad C=1$$

$$X(z) = Z \left[ \frac{1}{z-1} + \frac{1}{z^2-z+1} \right]$$

$$= \frac{Z}{z-1} + \frac{Z}{z^2-z+1}$$

$$\sin(\omega n) \xrightarrow{Z.T} \frac{Z \sin(\omega)}{z^2 - 2z \cos(\omega) + 1}$$

$$\cos(\omega n) \longrightarrow \frac{Z(z - \cos \omega)}{z^2 - 2z \cos \omega + 1}$$

$$\frac{\sin(\omega) Z}{(z^2 - z + 1) * \sin \omega} \Rightarrow \frac{Z \sin \omega}{z^2 - 2z \cos \omega + 1}$$

$$2 \cos(w) = 1 \Rightarrow \cos(w) = \frac{1}{2}$$

$$\therefore w = 60^\circ = \frac{\pi}{3} \text{ rad}$$

$$\sin w = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\frac{Z (\sqrt{3}/2)}{\frac{\sqrt{3}}{2} [Z^2 - Z + 1]}$$

$\rightarrow \sin(wn)$   
or  $\sin(\frac{\pi}{3}n)$

$$\Downarrow Z^{-1}T$$

$$= \frac{1}{\sqrt{3}/2} \sin(\frac{\pi}{3}n) = \frac{2}{\sqrt{3}} \sin(\frac{\pi}{3}n)$$

$$Z^{-1}T \Rightarrow x(n) = u(n) + \frac{2}{\sqrt{3}} \sin(\frac{\pi}{3}n)$$

← إذا طلب  $x(0)$ ,  $x(1)$ ,  $x(2)$  فمفوض عنه  
 $n$  بالقيم  $> 0$ .

[15] Lec 4

Ex  $y(n) - y(n-1] + 0.25 y(n-2) = u(n)$

For unit step i/p:

① Find  $y(0), y(1), y(2)$  using long division method.

② Find the unit step response  $y(n)$  using P.F.

Sol

Z.T  $Y(z) - z^{-1} Y(z) + 0.25 z^{-2} Y(z) = \frac{z}{z-1}$

where  $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\therefore u(z) = \frac{z}{z-1}$

$(1 - z^{-1} + 0.25 z^{-2}) Y(z) = \frac{z}{z-1}$

$Y(z) = \frac{z}{(z-1)(1 - z^{-1} + 0.25 z^{-2})} \quad * \frac{z^2}{z^2}$

$Y(z) = \frac{z^3}{(z-1)(z^2 - z + 0.25)}$



$$(Z-1)(Z^2 - Z + 0.25) = Z^3 - Z^2 + 0.25Z - Z^2 + Z - 0.25$$

$$= Z^3 - 2Z^2 + 1.25Z - 0.25$$

using long division

$$Y(z) = \frac{Z^3}{Z^3 - 2Z^2 + 1.25Z - 0.25}$$


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$$1 + 2Z^{-1} + 2.75Z^{-2}$$


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$$Z^3$$


---


$$-Z^3 + 2Z^2 + 1.25Z - 0.25$$


---


$$2Z^2 - 1.25Z + 0.25$$


---


$$-2Z^2 + 4Z + 2.5 - 0.5Z^{-1}$$


---


$$+2.75Z - 2.25 + 0.5Z^{-1}$$

$$y(0) = y(n)|_{n=0} = 1$$

$$y(1) = y(n)|_{n=1} = 2$$

$$y(2) = y(n)|_{n=2} = 2.75$$

□ T P 1

[2] using P.F

د. الى بنجيه  
فهو الى بنجيه فيه

$$Y(z) = \frac{z^3}{(z-1)(z^2-z+0.25)}$$

$$= \frac{z^3}{(z-1)(z-0.5)^2} = z \left[ \frac{z^2}{(z-1)(z-0.5)^2} \right]$$

$$= z \left[ \frac{A_1}{z-1} + \frac{A_2}{(z-0.5)^2} + \frac{A_3}{(z-0.5)} \right]$$

$$A_1 = \frac{1}{(0.5)^2} = 4 \quad , \quad A_2 = \frac{(0.5)^2}{(-0.5)} = -0.5$$

\* To find  $A_3$

Put  $z=0$

بما اني قيمه ل z بحيث  
لا تجعل المقام = 0

← وقارن الأجل بال P.F بـ

$$0 = -A_1 + 4A_2 - 2A_3 \Rightarrow A_3 = -3$$

$$\frac{z^2}{(z-1)(z-0.5)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z-0.5)^2} + \frac{A_3}{(z-0.5)}$$

نرى P.F

$$Y(z) = \frac{4z}{z-1} - \frac{0.5z}{(z-0.5)^2} - \frac{3z}{(z-0.5)}$$

$$n \xrightarrow{z.T} \frac{z}{(z-1)^2}$$

$$\frac{z}{(z-0.5)^2} = \frac{z}{(0.5)^2 \left(\frac{z}{0.5} - 1\right)^2} = \frac{z}{(0.25) \left(\frac{z}{0.5} - 1\right)^2}$$

$$= \frac{0.5}{(0.25)} \cdot \frac{z/0.5}{\left(\frac{z}{0.5} - 1\right)^2} \rightarrow a$$

$\downarrow$   
 $a$

$$a^n x(n) \xrightarrow{z.T} X\left(\frac{z}{a}\right)$$

$$\therefore X(z) = 4u(n) - \frac{z}{(0.5)^n} - 3(0.5)^n$$

$$a^n n \xrightarrow{z.T} \frac{z}{(z-1)^2} \Big|_{z=\frac{z}{a}}$$

$$= \frac{z/a}{\left(\frac{z}{a} - 1\right)^2} = \frac{az}{(z-a)^2}$$

Final value

$$y(\infty) = 4$$

EX  $X(z) = \frac{4z}{z^2 + 0.1z + 0.2}$

Find  $x(n)$

Sol

$$\sin(\omega n) \xrightarrow{Z.T} \frac{z \sin(\omega)}{z^2 - 2z \cos(\omega) + 1}$$

$$a^n \sin(\omega n) \rightarrow \frac{\frac{z}{a} \sin \omega}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right) \cos \omega + 1}$$

$$= \frac{a z \sin(\omega)}{z^2 - 2a z \cos \omega + a^2}$$

$$z^2 + 0.1z + 0.2 \equiv z^2 - 2a z \cos \omega + a^2$$

$$a^2 = 0.2 \Rightarrow a = \sqrt{0.2}$$

Ans. P. 1

$$\omega = 96 \cdot 4^\circ = 1.68 \text{ rad}$$

$$X(z) = \frac{4z(a \sin w)}{a \sin w [z^2 + 0.1z + 0.2]}$$

$$x(n) = \frac{4}{a \sin w} * a^n \sin(wn)$$

Report  $X(z) = \frac{z}{z^2 + 4}$

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